A PROBABILITY MODEL OF A PYRAMID SCHEME

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Periodically, the pyramid or "chain letter" scheme is offered to Americans under the guise of a business dealership. Recently, Glen Turner's Koscot Interplanetary Cosmetics firm has been charged with pyramiding by the FTC, SEC and various state agencies [2]. The total loss to the public has been estimated to be 44 million dollars. The promoters offer people a dealership or sales job in which most of their remuneration comes from recruiting new dealers (or salespersons). The basic fraud underlying a typical pyramid scheme is that every participant cannot recruit enough other people to recoup his investment, much less make a profit, since the pool of potential participants is soon exhausted.

The usual method of prosecuting such schemes is to show that if the representation of the promotional brochures were valid (e.g., members could recruit two new people a month), then within a short period of time (about 18 months) the entire population of the U.S. would have to participate. Thus, the last members would have no one to recruit. Although this argument based on the geometric progression is sometimes rejected by courts as unrealistic [3], pyramid scheme operators have placed a quota (or limit) on the number of participants in a specific geographic area in order to evade this line of prosecution. This article develops a probability model of this quota-pyramid scheme and the following results which also apply to unlimited schemes are derived:

1. The vast majority of participants have less than a 10% chance of recouping their initial investment when a small profit is achieved as soon as three people are recruited.

2. On the average, half of the participants will recruit no one else and lose all their money.

3. On the average, about one-eighth of the participants will recruit three or more people.

4. Less than one percent of the participants can expect to recruit six or more new participants.

While the above results can be "approximately" derived by ordinary limit theorems, for purposes of legal cases an absolute statement that a probability is small is more useful than an "approximate" statement. Thus, the above results are derived from a new probability bound on the sum of "small" binomial r.v.'s which is related to previous work of Hodges and LeCam [4].

Description of One Pyramid Scheme

A recent legal case in Connecticut [6] illustrates the confounding of legitimate business enterprise with a pyramid operation. People were offered dealerships in a "Golden Book of Values" for a fee of $2500. In return for their investment dealers could earn money in two ways: In each geographic area dealers were to develop a "Book of Values" for eventual sale to the public. First, they were to sell advertisements to merchants for $195 apiece and could keep half as a commission. Each advertisement offered a product or service at a discount, so that a "Book of Values" containing 50 to 100 discount offers could be sold to the public. The public was to pay $15 for the Book of Values, of which dealers were to keep $12. Second, a dealer had the right to recruit other dealers and was to receive $900 for each new recruit. Since the creation of a complete "Book of Values" for sale to the public takes a substantial amount of time, clearly the recruitment of new dealers is the most lucrative aspect of the venture.

In the recruitment brochure the possibility of earning large sums of money was illustrated by the following example: A dealer will bring people to weekly "Opportunity Meetings" and should be able to enroll other dealers at the rate of two per month. Thus, at the end of one year, the participant should receive $21,600 from the recruitment aspect alone. The prosecution showed that this misrepresents the earnings potential by asking the following question: "Suppose dealers who are enrolled can enroll two other dealers per month; as time went by, what would happen?" Professor Margolin (of Yale) testified that there would be a tripling of the number of dealers per month and by the end of 18 months, the geometric progression would exhaust the population of the United States. Clearly the cited recruitment brochure is misleading as all participants cannot come close to earning the indicated amount of money.

The "Golden Book of Values" pyramid system had an extra statistical nuance; i.e., there was a quota of 270 dealerships for the State of Connecticut. The Court noted that if each new dealer was successful in recruiting two dealers per month, only 27 would make a profit and the other 243 would lose money depending on how far down the pyramid they were.

Since a real pyramid operation would not be as regular as the Court described it, i.e., even at the beginning every participant would not enroll exactly two new dealers each month, in the next section we develop a probability model of the pyramid scheme. The model enables us to calculate the probability distribution of the number of people each participant will recruit and realize how strongly the probability of recouping one's initial investment depends on <u>when</u> the participant enters the pyramid scheme. Furthermore, the fraction of participants who can expect to recruit no one, can be derived.

<u>Calculating the Expected Return and Probability of Earning a Profit for Individual Participants in a Quota Pyramid System</u>

Economists evaluate the profitability of a business venture by comparing the initial investment to the "expected return" over a period of time. Suppose one is offered the opportunity to pay c dollars to enter a pyramid scheme which will terminate when the total number of participants is N where the fee for finding a new recruit is d dollars. Should one join? The answer is yes only if the expected number of people one will recruit, say R, is greater than c/d, i.e., one's expected earnings (Rd) are larger than the cost (c) of entering the plan. In this section we calculate the expected number of people the k^{th} participant will recruit assuming that all current participants have the same chance of recruiting the next member.

For ease in exposition we focus on the k^{th} entrant into the system. Since there are now k participants, each of whom presumably is recruiting, the probability that any particular one of the k current members recruits the next one is 1/k. Once the $k+1^{st}$ participant is recruited, each member has a chance of 1/(k+1) of recruiting the $k+2^{nd}$ participant, etc. Thus, the number of people the k^{th} participant will recruit is expressible as the sum of independent binomial r.v.'s,

$$S_k = \sum_{i=k}^{N-1} X_i, \qquad (3.1)$$

where each

$$X_i = \begin{cases} 1, & \text{with probability } p_i=1/i \\ 0, & \text{with probability } 1-1/i. \end{cases}$$

Thus, the expected number of people the k^{th} person will recruit equals

$$\sum_{i=k}^{N-1} 1/i \sim \ln[(N-\tfrac{1}{2}) / (k-\tfrac{1}{2})]. \qquad (3.2)$$

An immediate consequence of (3.2) is that once k is $\geq$ N/e, or about .37N, any future participant can expect to recruit no more than <u>one</u> person. Thus, only the 37% who join <u>first</u> can expect to recruit at least <u>one</u> new participant.

Another approach to demonstrating that a participant who joins the scheme after its initial phase has a <u>small</u> chance of recouping their investment is to calculate the probability that they will recruit the minimum number of people b = [c/d]+1, to achieve this. In our illustrative example, this value is 3. In order to compute $P(S_k \geq 3)$, statisticians use the Poisson approximation to the sum of binomials (3.1), as the p_i are small and decrease to zero. In the Appendix we describe a method of approximating S_k by Poisson r.v. P_k which is "stochastically larger" than S_k, and the probabilities presented in the table are derived from these results and are therefore upper bounds for the actual probabilities. (See table at the end of paper) The results in the table show that once a quota pyramid reaches <u>one-third</u> of its limit the probability a new member will regain his investment is less than 10%.

<u>The Expected Return to All Participants</u>

In the previous section we were concerned with the probability of each individual recruiting enough future members to regain the entrance fee. We now demonstrate that pyramid scheme investors are defrauded as a class.

The simplest proof of this is to notice that at any stage of the process (say K people are enrolled), the promoter (the first person) has received (K-1)c and has paid out (K-2)d. Hence, the promoter has a net profit of

$$c + (K-2)(c-d),$$

and the fraction of investment that has been returned to the participants is

$$\frac{K-2}{K-1}\frac{d}{c}.$$

Thus, the portion of all invested dollars returned to the participants is slightly less than d/c, the ratio of the fee earned for recruiting one new member to the initial investment. In the actual case used for illustration, this is only .36. Thus, as a class, participants will lose 64% of their investment.

Another interesting consequence of the

probability model is that on the average about half of the participants will recruit nobody and will lose their whole investment. This can be seen by noting that the probability that the k^{th} entrant will fail to recruit anyone is

$$P_k(0) = \prod_{i=k}^{N-1} (1-1/i) = (k-1)/(N-1).$$

Thus, the expected number of participants who are "shut out" is

$$\sum_{2}^{N} P_k(0) = \frac{1}{(N-1)} (1+\ldots+N-1) = \frac{N}{2},$$

i.e., half of the investors will lose everything they paid to join the system. Moreover, this remains true for any value of d (the amount paid for enrolling a new member). Thus, even if all the money paid in were returned to investors, half of them can expect to receive nothing.

One might question the relevance of the previous result in the context of a fraud case if a significant fraction of the participants were "big winners". When we replace the r.v. S_k denoting the number of people the k^{th} entrant recruits by its Poisson majorizer P_k, one can show (see Appendix) that as $N \to \infty$, the proportion of the participants who recruit exactly r people approaches $2^{-(r+1)}$ so that the fraction who recruit at least r is 2^{-r}. Thus only one-eighth of the participants can expect to recruit at least three members, and only one in 16 million can expect to recruit 24 or more people. Thus, our model agrees with the findings of Judge Naruk in the case described when he noted that no one had earned an amount of money near that claimed in the brochure.

In light of this and other facts, Judge Naruk permanently enjoined the defendants from selling or authorizing others to sell Goldren Book dealerships and from instituting any other multi-level merchandising plan in Connecticut without express court approval.

Appendix: A Probability Bound for the Sum of Poisson-Binomial Variates

Let X_i, $i=1,\ldots,n$, be independent binomial r.v.'s with $p_i=P(X_i=1)$ and let $S=\Sigma X_i$. When the probabilities p_i are "small", we desire a tight upper bound rather than an approximation to

$$P\{\sum_{i=1}^{n} X_i > a\}, \quad (1)$$

where a is a specified integer usually greater than the expected value, Σp_i, of the r.v.'s.

In order to derive a bound for (1), we introduce Poisson r.v.'s Y_i, which are stochastically larger than the X_i's, i.e. we choose the parameter λ_i of Y_i to satisfy

$$P(Y_i=0) = P(X_i=0) = 1-p_i, \quad (2)$$

i.e.,

$$e^{-\lambda_i} = 1-p_i \text{ or } \lambda_i = -\ln(1-p_i). \quad (3)$$

In order to give X_i and Y_i a bona fide joint distribution, following Hodges and LeCam, we define

$$P(X_i=0,\ Y_i=0) = 1-p_i$$

and

$$P(X_i=1,\ Y_i=k) = \frac{e^{-\lambda_i}\lambda_i^k}{k!} \quad (4)$$

where λ_i and p_i obey (3).

As $X_i \leq Y_i$ for each i, $\Sigma X_i \leq \Sigma Y_i$ and

$$P(\Sigma X_i \geq a) \leq P(\Sigma Y_i \geq a). \quad (5)$$

As ΣY_i has a Poisson distribution, the probability on the right is readily computable once λ_i is expressed in terms of p_i. From the Taylor expansion,

$$-\ln(1-x) = \sum_{1}^{\infty} x^j/j,$$

it follows that

$$\sum_{j=1}^{k} \frac{x^j}{j} \leq -\ln(1-x) \leq \sum_{j=1}^{k-1} \frac{x^j}{j} + \frac{x^k}{k}\frac{1}{(1-x)} \quad (6)$$

so each λ_i can be obtained to any desired accuracy. For practical purposes, the choice of k=3 usually suffices, so (6) becomes

$$p_i+\frac{p_i^2}{2}+\frac{p_i^3}{3} \leq \lambda_i \leq p_i + \frac{p_i^2}{2}+\frac{p_i^3}{3}\frac{1}{(1-p_i)}.$$

When the $\{p_i\}$ decrease, the difference between the bounds on the parameter

$$\sum_{j}^{N} \lambda_i$$

of the Poisson r.v. majorizing S is

$$[(1-p_j)^{-1}-1][\sum_{i=j}^{N} p_i^3]/3.$$

Before applying the above method to our special case we present the analog of Hodges and LeCam's results for the difference between $P(S \geq a)$ and our approximation P_λ. Specifically, we have

Lemma: For any constant a,

$$P(P_\lambda \geq a) - P(S \geq a) \leq \Sigma p_i^2 \quad (7)$$

Proof: For each i,

$$P(Y_i > X_i) = P(Y_i \neq X_i) = \sum_{k=2}^{\infty} e^{-\lambda_i} \frac{\lambda_i^k}{k!} =$$

$$1-e^{-\lambda_i} - \lambda_i e^{-\lambda_i} = p_i + (1-p_i)\ln(1-p_i).$$

As $e^{-x} \geq 1-x$, $-x \geq \ln(1-x)$, so

$$p_i + (1-p_i)\ln(1-p_i) \leq p_i - (1-p_i)p_i = p_i^2.$$

By Boole's inequality, $P(P_\lambda > S) \leq \Sigma p_i^2$.

Application to the Pyramid Scheme

In our example, $p_i = 1/i$, and we desire to approximate

$$S_k = \sum_{i=k}^{N=1} X_i$$

by a Poisson variable.

In our case we can obtain an explicit expression for γ_k rather than using a Taylor series development as

$$\lambda_i = -\ln(1-1/i) = \ln(i/i-1).$$

Thus,

$$\gamma_k = \sum_{i=k}^{N-1} \lambda_i = \sum_{k}^{N-1} [\ln i - \ln(i-1)] = \ln((N-1)/(k-1))$$

so that

$$P(S_k \geq r) \leq \sum_{i=r}^{\infty} e^{-\gamma_k} \frac{\gamma_k^i}{i!} = 1 - \sum_{i=1}^{r-1} \frac{(k-1)}{(N-1)} \frac{\gamma_k^i}{i!}. \quad (8)$$

Since S_k is the sum of non-identically distributed binomial r.v.'s, a compact formula for its exact distribution is not available and a computer is needed. In Table A.1, we compare the exact value of $P(S_k \geq 2)$ and $P(S_k \geq 3)$ to the bounds we obtained from formula (8). Clearly the bounds are quite close.

As our r.v.'s P_k approximating S_k are so close we can derive an accurate approximation to the expected fraction of participants who will recruit at least r people. Formally, we have

Theorem: Let $X_2, X_3, \ldots, X_N$, be a sequence of Poisson r.v.'s with parameters

$$\gamma_k = [\ln(\frac{N-1}{k-1})].$$

Then

$$\frac{1}{(N-1)} \sum_{k=2}^{N-1} P(X_k = r) \to 2^{-(r+1)}, \quad r=0,1,2,\ldots \quad (9)$$

as $N \to \infty$.

Proof: As $P(X_k = r) = \gamma_k^r e^{-\gamma_k}/r!$, the left side of (9) is

$$\frac{1}{r!} (N-1)^{-1} \sum_{2}^{N-1} \frac{(k-1)}{(N-1)} [\ln(\frac{N-1}{k-1})]^r. \quad (10)$$

Letting $v = (k-1)/(N-1)$, (10) is a Riemann approximation to

$$(r!)^{-1} \int_0^1 v[\ln(\frac{1}{v})]^r dv =$$

$$(r!)^{-1} \int_0^{\infty} z^r e^{-2z} dz = 2^{-(r+1)}.$$

Hence, for large N, the expected fraction of all participants who recruit at least r people is $1/2^r$ for $r=0,1,2,\ldots$

In order to see how fast the limit is approached we computed the exact values of (9) when N=270 and 1000 for r=1,2, and 3. The resulting values which have limits 1/4, 1/8, 1/16, were .24991, .12475 and .06202 (N=270) and .24999, .12497, .06244 (N=1000)

References

[1] Anderson, T.W., and Samuels, S.M. (1965) Some Inequalities Among Binomial and Poisson Probabilities. Fifth Berkeley Symposium. Vol. I. pp. 1-12.

[2] Feore, J. (1974) Pyramid Scheme: Dare to be Regulated. Georgetown Law Journal. pp. 1257-1293.

[3] Ger-ro-Mar. Inc. vs. F.T.C. (1975) 518 F 2d 33.

[4] Hodges, J.L., Jr., and LeCam, L. (1960) The Poisson Approximation to the Poisson Binomial Distribution. Ann. Math. Statistics. Vol. 31, No. 3 pp. 737-740.

[5] Hoeffding, W. (1956) On the Distribution of the Number of Successes in Independent Trials. Ann. Math Statistics. Vol. 36. pp. 713-721.

[6] Naruk, H. (1975) Memorandum of Decision: State of Connecticut vs. Bull Investment Group. 32 Conn, Sup. 279.

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Addenda: The Case in Which the Court Rejected the "Geometric Progression" Argument

In order to motivate the development of our model, this addenda will describe the decision in the Ger-Ro-Mar vs. F.T.C. case [3]. The company manufactured and sold brassieres, girdles, swimwear and lingerie under the label Symbra'Ette. Its sales grew from $37,000 in 1965 to $2 million in 1969 but fell to $1.2 million in 1972. The company sold its products through a sales force which required distributors to buy an inventory of products in order to participate. The sales organization was a multi-level one in which supervisors earned a percentage of the sales of those below them. The entry level (Key Distributor) required a purchase of $300 retail value of merchandise (i.e. an initial investment of $215, the wholesale value). A Key Distributor sold the products door to door and could also engage in unlimited recruiting and become a Senior Key Distributor when the retail value of the merchandise he and his recruits reached $1000 in any month. The profit for Senior Keys consisted of an increased profit margin on their own retail sales and a percentage profit on the purchases made by his recruits and various related commissions. Similarly, Senior Keys could rise to higher levels when they and their recruits achieved the requisite retail value of products purchased from the company.

To induce individuals to participate in the program the promotional brochure illustrated how, both by building a large personal group of sales people via recruitment and by selling at retail, a person could earn large sums of money, e.g. $56,400 per year as a District Manager.

Before quoting the 6th Circuit's decision, it should be noted although there is a pyramiding aspect in this program, the situation differs from the Golden Book case because

1) the initial investment was relatively small ($215) and

2) since there was a product available to sell one did not have to recruit others to rise in the "system" in order to earn money.

We now quote from the decision:

"The sole evidence to support the Commission's holding that the plan is inherently unfair and deceptive is a mathematical formula, which shows that if each participant in the plan recruited only five new recruits each month and each of those in turn recruited five additional recruits in the following month, and this process were allowed to continue, at the end of only 12 months the number of participants would exceed 244 million including presumably the entire staff of the FTC. The Commission concludes that this, in effect, is the impossible dream and that the siren song of Symbra'Ette must be stilled. We find no flaw in the mathematics or the extrapolation and agree that the prospect of a quarter of a billion brassiere and girdle hawkers is not only impossible but frightening to contemplate, particularly since it is in excess of the present population of the Nation, only about half of whom hopefully are prospective lingerie consumers. However, we live in a real world and not fantasyland.

As indicated by the record, the fact is that Symbra'Ette, which commenced business in 1963, did not reach its peak in distributorships until 1972 when it had attracted some 3,635 distributors. The record does not indicate the geographical distribution of these vendors, and we have no study or analysis in the record which would realistically establish that some recruiting saturation exists which would make the entry of additonal distributors and the recruitment of others potentially impossible in any practical sense. While the Commission need not establish actual deception by the testimony of disappointed entrepreneurs, it has failed entirely to establish a potential threat. Not all Americans aspire to the calling in issue and not all who are attracted will continue indefinitely."

Apparently the F.T.C. relied solely on the "geometric progression" argument rather than obtaining data to estimate the crucial statistical quantities such as the average number of new recruits each participant achieves or the proportion of sales persons who recruit no one else. Without such supporting evidence, it is difficult to convince a Court that it is mathematically impossible for a business to survive when it has existed for a number of years. Although the author hasn't seen the company's books, he feels that they would show that a substantial portion of the merchandise sold was to new recruits rather than to the public at retail.

Table 1 The Expected Number of People Each Participant will Recruit and Upper Bounds for the Probability of Recruiting at Least 2 or 3 New Members (N = 270)

Position of Entry	Expected No. of Recruits	Probability of Recruiting at Least r New Members r=2	r=3
k= 5	4.208	.9226	.7909
10	3.398	.8529	.6598
20	2.6500	.7422	.4941
30	2.227	.6521	.3846
40	1.931	.5750	.3047
50	1.703	.5077	.2435
60	1.517	.4479	.1955
75	1.291	.3699	.1407
90	1.106	.3032	.1008
100	1.1000	.2641	.0802
120	.8160	.1968	.0497
135	.697	.1547	.0338
150	.591	.1189	.0222
180	.407	.0635	.0083
210	.2524	.0270	.0022
240	.1182	.0065	.0003

Table A.1 Comparison of Our Bounds to the Exact Values (N=270)

Index (k)	Exact $P(S_k \geq 2)$	UB for $P(S_k \geq 2)$	Exact $P(S_k \geq 3)$	UB for $P(S_k \geq 3)$
5	.92064	.92255	.78289	.79087
10	.85104	.85287	.65358	.65978
25	.69346	.69517	.43053	.43464
40	.57344	.57504	.30162	.30470
50	.50613	.50765	.24095	.24354
75	.36851	.36986	.13900	.14074
90	.30195	.30319	.09943	.10081
135	.15378	.15472	.03311	.03376
180	.06290	.06353	.00807	.00833
210	.02655	.02696	.00212	.00222
240	.00626	.00646	.00023	.00025

Remark: The ordinary Poisson approximation also yields precise estimates of the exact probabilities. These estimates are slightly low for the early values of k and become slightly high for larger values of k as would be expected from the literature [1] on the Poisson approximation to the sum of i.i.d. binomial r.v.'s.